Photon statistics and squeezing properties in a wiggler-free free-electron laser

1993 J. Phys. A: Math. Gen. 26 5107

(http://iopscience.iop.org/0305-4470/26/19/038)

J. Phys. A: Math. Gen. 26 (1993) 5107–5113. Printed in the UK

Photon statistics and squeezing properties in a wiggler-free free-electron laser

Gou San-kui

CCAST (World Laboratory) P.O. Box 8730, Beijing 100080, People's Republic of China and Department of Physics, Lanzhou University, Lanzhou, Gansu 730000, People's Republic Of China

Received 15 March 1993, in final form 6 July 1993

Abstract. In a particular frame moving at the electron velocity along the axial direction, photon statistics and squeezing properties are discussed for a wiggler-free free-electron laser under the weak-coupling and small electron-recoil limits. It is shown that a wiggler-free free-electron laser is more powerful than a wiggler-pumped free-electron laser for conventional experimental parameters.

Cyclotron radiation by relativistic electrons in an external static and uniform magnetic field can be generated in two kinds of devices: gyrotron [1, 2] and wiggler-free free-electron laser [3–5]. The electrons spiral on a helix in both devices, and the main difference between the two kinds of devices is that the perpendicular electron velocity is much larger than the parallel component of electron velocity in the gyrotron, but the opposite is true in the wiggler-free free-electron laser. Recently, Zhang has shown, through a classical manipulation [6], that Madey's theorem [7] is valid for a gyrotron. Fu and Lee [8] have given the spherical Raman–Nath equation for gyrotron radiation starting out with the induced dipole moment. Moreover, Šoln [5] has obtained the quantum mechanical gain of the wiggler-free free-electron laser. In this short paper, analogous to the quantum approach in the wiggler-pumped free-electron laser [9–14], we give the spherical Raman–Nath equation for, and discuss the photon statistics and squeezing properties in, a wigger-free free-electron laser.

In the laboratory frame, we consider a beam of axial relativistic electrons propagating through an axial uniform magnetic field and a radiation field is induced. The vector potentials of these fields are

$$A_0=\frac{\sqrt{2}}{4}B_0[-e(y+\mathrm{i}x)+e^*(-y+\mathrm{i}x)] \tag{1}$$

$$A_r=A_r^*e\,\mathrm{e}^{-\mathrm{i}(K_rz-\omega_rt)}+A_r\,e^*\,\mathrm{e}^{\mathrm{i}(K_rz-\omega_rt)} \tag{2}$$

where

$$e=\frac{e_x+\mathrm{i}e_y}{\sqrt{2}}\qquad e^*=\frac{e_x-\mathrm{i}e_y}{\sqrt{2}},$$

0305-4470/93/195107+07 \$07.50

K_r and ω_r are the wavenumber and frequency of the radiation field, respectively. The single-electron Hamiltonian is

$$H=[m_0^2c^4+c^2(P-eA)^2]^{1/2} \tag{3}$$

where c is the speed of light in vacuum, m_0 is the rest mass of the electron, e is the electron charge, $P=(P_x, P_y, P_z)$ is the canonical momentum of the electron, and $A=A_0+A_r$. After performing a Lorentz transformation from the laboratory frame to a particular frame moving at the electron velocity v_{z0} along the axial direction, one can conclude that the electron motion is non-relativistic in the moving frame and obtain

$$x'=x \tag{4}$$

$$y'=y \tag{5}$$

$$z'=\gamma_0(z-v_{z0}t) \tag{6}$$

$$t'=\gamma_0\left(t-\frac{v_{z0}}{c^2}z\right) \tag{7}$$

$$k'=\gamma_0\left(K_r-\frac{v_{z0}}{c^2}\omega_r\right) \tag{8}$$

$$\omega'=\gamma_0(\omega_r-v_{z0}K_r) \tag{9}$$

where $\gamma_0=1-(v_{z0}^2/c^2)]^{-1/2}$, and the primes represent the physical quantities in the moving frame. Then the single-electron Hamiltonian is

$$\begin{aligned} H'&=[m_0^2c^4+c^2(P'-eA')^2]^{1/2} \\ &\cong mc^2+\frac{P'^2}{2m}-\frac{e(A_0'+A_r')\cdot P'}{m}+\frac{e^2A_0'^2}{2m}+\frac{e^2(A_r'\cdot A'_0)}{m} \end{aligned} \tag{10}$$

where $m=[m_0^2+(e^2A_r'^2/c^2)]^{1/2}$. As in [9–11], one can quantize the laser field through the following transformations

$$A_r\rightarrow e^{i\omega't'}\left[\frac{\hbar}{2\omega'\varepsilon_0V}\right]^{1/2}a_r \tag{11}$$

$$A_r^*\rightarrow e^{-i\omega't'}\left[\frac{\hbar}{2\omega'\varepsilon_0V}\right]^{1/2}a_r^+ \tag{12}$$

and quantize the electron transverse motion through the commutator $[x_\alpha', P_\beta']=i\hbar\delta_{\alpha\beta}$, then obtain the Hamiltonian for the electron–field system in the moving frame

$$\begin{aligned} H=&mc^2+\frac{\hbar}{2}(\omega+\omega_c)+\frac{p_z^2}{2m}+\hbar\omega a_r^+a_r+\hbar\omega_cT_+T_- \\ &-\hbar(\omega_c\Omega)^{1/2}[a_r^+T_-e^{-ikz}+a_rT_+e^{ikz}] \end{aligned} \tag{13}$$

where the primes are dropped from now on with the understanding that all physical quantities refer to the moving frame; $\omega_c=eB_0/m$ is the cyclotron frequency, $\Omega=$

$e^2/2m\omega\varepsilon_0 V$ is the coupling constant, and the electron transverse motion is described by the operators

$$T_+ = \frac{\left[(\hat{P}_x - \mathrm{i}\hat{P}_y) + \frac{ieB_0}{2}(x - \mathrm{i}y)\right]}{\sqrt{2m\hbar\omega_c}} \tag{14}$$

$$T_- = \frac{\left[(\hat{P}_x + \mathrm{i}\hat{P}_y) - \frac{\mathrm{i}eB_0}{2}(x + \mathrm{i}y)\right]}{\sqrt{2m\hbar\omega_c}} \tag{15}$$

with the commutator $[T_-, T_+] = 1$. It is easy to obtain the three constants of motion from the Hamiltonian (13)

$$a_r^+ a_r + T_+ T_- = \text{constant} \tag{16}$$

$$\hat{p}_z + \frac{\hbar k}{2}(a_r^+ a_r - T_+ T_-) = \text{constant} \tag{17}$$

$$x\hat{P}_y - y\hat{P}_x - \hbar a_r^+ a_r = \text{constant.} \tag{18}$$

The above formulas represent the conservation of the quantum number, axial linear momentum, and angular momentum of the system, respectively.

If we start from an initial state with electron axial momentum P_0, and the electron transverse motion and laser field are in coherent states with a mean number of quanta $|\alpha_{T0}|^2$ and photons $|\alpha_{r0}|^2$, respectively, then the state at any time t can be written as

$$|\Psi_{(t)}\rangle = \sum_{n_T=0}^{\infty} \sum_{n_r=0}^{\infty} \sum_{l=-n_r}^{n_T} C_{l(t)}^{n_T, n_r} \exp\left[-\frac{\mathrm{i}t}{\hbar}\left(mc^2 + \frac{\hbar\omega}{2} + \frac{\hbar\omega_c}{2} + n_r\hbar\omega + n_T\hbar\omega_c + \frac{P_0^2}{2m}\right)\right]$$
$$\times \frac{\exp[-\frac{1}{2}(|\alpha_{T0}|^2 + |\alpha_{r0}|^2)]}{\sqrt{n_T! n_r!}} \alpha_{T0}^{n_T} \alpha_{r0}^{n_r} |P_0 - l\hbar k; n_T - l; n_r + l\rangle \tag{19}$$

where l is an integer enumerating the emitted photons, and $C_{l(t)}^{n_T, n_r}$ is the probability amplitude for interchanging l photons in the presence of n_T electron transverse quanta and n_r laser photons, respectively. Substituting (13) and (19) into Schrödinger's equation

$$\mathrm{i}\hbar \frac{\partial}{\partial t} |\psi_{(t)}\rangle = H |\psi_{(t)}\rangle \tag{20}$$

we obtain the spherical Raman–Nath equation for a wiggler-free free-electron laser

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} C_{l(t)}^{n_T, n_r} = (\mu l + \nu l^2) C_{l(t)}^{n_T, n_r}$$
$$-\lambda\left[\sqrt{(n_r + l + 1)(n_T - l)}\, C_{l+1(t)}^{n_T, n_r} + \sqrt{(n_r + l)(n_T - l + 1)}\, C_{l-1(t)}^{n_T, n_r}\right] \tag{21}$$

where $\lambda = (\omega_c \Omega)^{1/2}$, $\mu = \omega - \omega_c - (kP_0/m)$, $\nu = \hbar k^2/2m$. The above spherical Raman–Nath equation is analogous to that well known in the wiggler-pumped free-electron laser [9–14]. To simplify further discussion, we assume that the initial laser field is in vacuum, that is $\alpha_{r0} = 0$. Following the procedure as in [12–14], under the limit $|\mu| \gg \nu$, $\Omega \to 0$, $\alpha_{T0} \to \infty$, with $\lambda \alpha_{T0} =$ constant $\bar{\Omega}$, we can obtain the average photon number

$$\langle l \rangle = \langle \psi_{(t)} | a_r^+ a_r | \psi_{(t)} \rangle = \bar{\Omega}^2 t^2 \left(\frac{\sin\theta}{\theta} \right)^2 - \frac{\nu \bar{\Omega}^2 t^3}{2} \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\frac{\sin\theta}{\theta} \right)^2 \tag{22}$$

the gain

$$G = \langle l \rangle - \langle l \rangle |_{\nu=0} = -\frac{\nu \bar{\Omega}^2 t^3}{2} \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\frac{\sin\theta}{\theta} \right)^2 \tag{23}$$

the photon statistics parameter [15]

$$\Delta = \langle \psi_{(t)} | (a_r^+ a_r)^2 | \psi_{(t)} \rangle - \langle \psi_{(t)} | a_r^+ a_r | \psi_{(t)} \rangle^2 - \langle \psi_{(t)} | a_r^+ a_r | \psi_{(t)} \rangle = -\frac{\nu \bar{\Omega}^4 t^5}{2} \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\frac{\sin\theta}{\theta} \right)^4 \tag{24}$$

and the squeezing parameter of the electron

$$(\Delta \hat{P}_z)^2 = \langle \psi_{(t)} | \hat{P}_z^2 | \psi_{(t)} \rangle - \langle \psi_{(t)} | \hat{P}_z | \psi_{(t)} \rangle^2 = \hbar^2 k^2 \bar{\Omega}^2 t^2 \left(\frac{\sin\theta}{\theta} \right)^2 \tag{25}$$

where $\theta = -\mu t/2$. Formula (23) means that Madey's theorem [7] is also valid for a wiggler-free free-electron laser. The gain is proportional to the quanta of electron-transverse motion and its maximum occurs at $\theta_{\max} = 1.3$. In the laboratory-frame variables, the laser frequency corresponding to the maximum-gain condition is

$$\omega_r = \gamma_0 \left(1 + \frac{v_{z0}}{c} \right) \left[\omega_c - \frac{2.6 \gamma_0 v_{z0}}{L} \right]$$

where L is the interaction length in the laboratory frame. So the laser frequency is Doppler up-shifted in the relativistic case. For example, for $\gamma_0 = 3$ (or 4), $v_{z0} \lessapprox c$, $B_0 = 1$ (or 1.5)T, and $L = 100$ (or 50) cm, we have $\omega_r \cong 2\gamma_0 \omega_c = 3.5 \times 10^{11}$ (or 7×10^{11}) Hz; these are in the millimetre-wave region. However, in the non-relativistic case, $v_{z0} \ll c$, $\gamma_0 \simeq 1$, we have $\omega_r \cong \omega_c - (2.6 v_{z0}/L)$; these are in the centimetre-wave region for the same guide-field magnitudes given above.

There exists a critical value of relativistic factor $\gamma_c = \omega_r / 2\omega_c$ for a wiggler-free free electron laser. The positive gain corresponds to super Poissonian (bunching) for $\gamma_0 > \gamma_c$ (or $\theta > 0$), the negative gain corresponds to sub-Poissonian (antibunching) for $\gamma_0 < \gamma_c$ (or $\theta < 0$), and the zero gain corresponds to the maximum spontaneous emission and Poissonian for $\gamma_0 = \gamma_c$ (or $\theta = 0$). Similar, well-known results have been obtained for a gyrotron where the laser frequency is Doppler down-shifted and the critical value of the relativistic factor in the transverse dimension is $\gamma_c = \omega_c / \omega_r$. The relation between the sign of the gain and photon statistics in a wiggler-free free–electron laser can be explained following the physical argument given by Becker and McIver [9] for a wiggler-pumped free-electron laser. By analogy, we can understand formula (25). If an electron emits a laser photon, the electron momentum changes $\hbar k$, or, emitting one laser photon would cause the uncertainty $\hbar^2 k^2$ in the electron momentum. So the average electron spread should be the uncertainty of the electron momentum caused by emitting a laser photon, times the probability of emitting one laser photon (or spontaneous emission).

The ratio between the maximum gains of the wiggler-free and wiggler-pumped [9–14] free-electron lasers is

$$\xi = \frac{\text{maximum gain of wiggler-free free electron laser}}{\text{maximum gain of wiggler-pumped free electron laser}}$$

$$= \omega_c |\alpha_{T0}|^2 / 4Nw\Omega \tag{26}$$

where N_w represents the strength of the wiggler field. In the laboratory-frame variables, we have

$$\hbar N_w \Omega = \frac{e^2 B_w^2}{2mK_w^2} \tag{27}$$

where K_w is the wavenumber of the wiggler and B_w is the strength of the wiggler. Suppose the initial transverse energy $\hbar\omega_c|\alpha_{T0}|^2$ of the electron motion be the non-relativistic classical equilibrium energy $\frac{1}{2}mV_\perp^2$ of the electron vertical motion, approximately, then one can obtain

$$\xi = \frac{V_\perp^2}{\left(\dfrac{2eB_w}{mK_w}\right)^2}. \tag{28}$$

For conventional experimental data, $V_\perp/c \approx 10\%$, $B_w \approx 0.05$–$0.5\,\mathrm{T}$, $\lambda_w = 2\pi/K_w \approx$ 2–10 cm, and $eB_w/mcK_w \approx 5\times10^{-2}$ or less, so we have

$$\xi \gg 1. \tag{29}$$

The wiggler-free free-electron laser is more powerful than the wiggler-pumped free electron laser.

Finally, we give some discussion on the nonlinear nature of the spherical Raman–Nath equation (21) caused by the term νl^2. Following the procedure given in [16–19], one can derive the equation (10) of [20]. If the Lie elements defined in these references were constant operators, we could solve the differential-difference equation (21) through the transformation (11) of [20]. Recently, it has been found that these Lie generators are not invariant, so such a procedure needs to be amended. In principle, the spherical Raman–Nath equation can be solved using the Baker–Hausdorff formula. First, we make the transformation

$$C_{l(t)}^{n_T, n_r} = (-\mathrm{i})^l \, e^{-\mathrm{i}t(n_T - n_r)[2\mu + \nu(n_T - n_r)]/4} |M_{l(t)}^{n_T, n_r}\rangle \tag{30}$$

and define the Lie operators

$$\hat{J}_+ |M_{l(t)}^{n_T, n_r}\rangle = \sqrt{(n_r + l + 1)(n_T - l)} \, |M_{l+1(t)}^{n_T, n_r}\rangle \tag{31}$$

$$\hat{J}_- |M_{l(t)}^{n_T, n_r}\rangle = \sqrt{(n_r + l)(n_T - l + 1)} \, |M_{l-1(t)}^{n_T, n_r}\rangle \tag{32}$$

$$\hat{J}_z |M_{l(t)}^{n_T, n_r}\rangle = \left[l + \frac{(n_r - n_T)}{2} \right] |M_{l(t)}^{n_T, n_r}\rangle \tag{33}$$

with the commutation relations

$$[\hat{J}_+, \hat{J}_-] = 2\hat{J}_z \tag{34}$$

$$[\hat{J}_z, \hat{J}_\pm] = \pm \hat{J}_\pm . \tag{35}$$

Then, the equation (21) changes to

$$\frac{\mathrm{d}}{\mathrm{d}t} |M_{l(t)}^{n_T, n_r}\rangle = \{ -\mathrm{i}\nu \hat{J}_z^2 - \mathrm{i}[\mu + \nu(n_T - n_r)]\hat{J}_z + \lambda \hat{J}_+ - \lambda \hat{J}_- \} |M_{l(t)}^{n_T, n_r}\rangle . \tag{36}$$

Using (31)–(36), one can obtain

$$\frac{\mathrm{d}\hat{J}_+}{\mathrm{d}t} = 2\lambda \hat{J}_z - \mathrm{i}[\mu + \nu(n_T - n_r)]\hat{J}_+ - \mathrm{i}\nu(\hat{J}_+ \hat{J}_z + \hat{J}_z \hat{J}_+) \tag{37}$$

$$\frac{\mathrm{d}\hat{J}_-}{\mathrm{d}t} = 2\lambda \hat{J}_z + \mathrm{i}[\mu + \nu(n_T - n_r)]\hat{J}_- + \mathrm{i}\nu(\hat{J}_- \hat{J}_z + \hat{J}_z \hat{J}_-) \tag{38}$$

$$\frac{\mathrm{d}\hat{J}_z}{\mathrm{d}t} = -\lambda(\hat{J}_+ + \hat{J}_-) . \tag{39}$$

It is easy to obtain an invariant operator from (37)–(39)

$$\lambda(\hat{J}_+ - \hat{J}_-) - \mathrm{i}\nu \hat{J}_z^2 - \mathrm{i}[\mu + \nu(n_T - n_r)]\hat{J}_z = \text{constant operator.} \tag{40}$$

So the solution of (36) is

$$|M_{l(t)}^{n_T, n_r}\rangle = \exp\{ -\mathrm{i}\nu t \hat{J}_z^2 - \mathrm{i}t[\mu + \nu(n_T - n_r)]\hat{J}_z + \lambda t \hat{J}_+ - \lambda t \hat{J}_- \} |M_{l(0)}^{n_T, n_r}\rangle . \tag{41}$$

The next step is to expand the right-hand side of (41) using the Baker–Hausdorff formula. However, we do not carry out the tedious manipulation here.

Acknowledgments

This work is supported by the Chinese National Nature Science Foundation and Gansu Nature Science Foundation of China.

References

[1] Schneider J 1959 *Phys. Rev. Lett.* 2 504
[2] Hirshfield J L 1979 *Infrared and Millimeter Waves* vol 1 ed K J Button (New York, Academic)
[3] Ride S K and Colson W B 1979 *Appl. Phys.* 20 41

[4] Fruchtman A and Friedland L 1982 *J. Appl. Phys.* **53** 4011
[5] Šoln J 1985 *J. Appl. Phys.* **58** 3314
[6] Zhang S C 1989 *Ke Xue Tong Bao* (China) **34** 665
[7] Madey J M J 1979 *Nuovo Cimento* B **50** 64
[8] Fu Q Y and Lee C T 1990 *J. Phys. A: Maths. Gen.* **23** L99
[9] Becker W and McIver J K 1983 *Phys. Rev.* A **27** 1030
[10] Orszag M and Ramirez R 1986 *J. Opt. Soc. Am.* B **3** 895
[11] Orszag M 1987 *Phys. Rev.* A **36** 189
[12] Dattoli G, Gallardo J, Renieri A and Richetta M 1985 *IEEE J. Quantum Electron.* **QE-21** 1069
[13] Dattoli G, Gallardo J and Torre A 1986 *J. Opt. Soc. Am.* B **3** 65
[14] Dattoli G, Gallardo J and Torre A 1987 *J. Opt. Soc. Am.* B **4** 185
[15] Stoler D 1974 *Phys. Rev. Lett.* **33** 1379
[16] Bosco P, Dattoli G and Richetta M 1984 *J. Phys. A: Maths. Gen.* **17** L395
[17] Ciocci F, Dattoli G and Richetta M 1984 *J. Phys. A: Maths. Gen.* **17** 1333
[18] Bosco P, Gallardo J and Dattoli G 1984 *J. Phys. A: Maths. Gen.* **17** 2739
[19] Dattoli G, Gallardo J and Torre A 1986 *J. Math. Phys.* **27** 772
[20] Gou S K 1989 *J. Phys. A: Maths. Gen.* **22** 2653. Note that the two terms $e^{(\pm i\gamma \mathcal{L}_{\pm}^{2})}$ on the right hand side of the equation (10) should be $e^{(\pm i\gamma\chi \mathcal{L}_{\pm}^{2})}$, respectively.